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New Application of Noncanonical Maps in Quantum Mechanics

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One invertible and one unitary operator can be used to reproduce the effect of a q-deformed commutator of annihilation and creation operators. The original annihilation and creation operators are mapped into new operators, not conjugate to each other, whose standard commutator equals the identity plus a correction proportional to the original number operator. The consistency condition for the existence of this new set of operators is derived, by exploiting the Stone theorem on 1-parameter unitary groups. The above scheme leads to modified "equations of motion" which do not preserve the properties of the original first-order set for annihilation and creation operators. Their relation with commutation relations is also studied.

KEY WORDS: harmonic oscillator; deformation quantization; 1-parameter unitary groups.

1. INTRODUCTION

Several efforts have been devoted in the literature to the attempt of building quantum mechanics as a kind of deformed classical mechanics. The mathematical foundations and the physical applications of such a program are well described, for example in Sternheimer (1998) and in the many references given therein. Within that framework, quantization emerges as an autonomous theory based on a deformation of the composition law of classical observables, not on a radical change in the nature of the observables. One then gets a more general approach which coincides with the conventional operatorial approach in known applications whenever a Weyl map can be defined, and leads to an improved conventional quantization in field theory (Sternheimer, 1998).

In particular, this has led to consider the socalled q-deformed commutator of annihilation and creation operators of an harmonic oscillator, i.e. (Arik and

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Coon, 1976)

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = I, \tag{1.1}$$

I being the identity operator. The aim of this study was to provide an alternative interpretation of Eq. (1.1) and discuss its implications, putting instead the emphasis on maps which do not preserve the canonical commutation relations. In other words, since noncanonical maps are an important topic in quantum mechanics, we propose to exploit their properties to avoid having to deform the composition law of observables. The following sections show under which conditions this is indeed possible, and some of its implications.

2. A NEW LOOK AT DEFORMED COMMUTATORS

We first point out that Eq. (1.1) can be reexpressed in the form

$$aa^\dagger - a^\dagger a = I + (q-1)a^\dagger a. \tag{2.1}$$

The left-hand side of Eq. (2.1) is the application to a and $a^\dagger$ of the standard definition of commutator of a pair of linear operators A and B:

$$[A, B] \equiv AB - BA, \tag{2.2}$$

where, at this stage, we are leaving aside the technical problems resulting from the possible occurrence of unbounded operators (Prugovecki, 1981; Reed and Simon, 1972).

The picture we have in mind is therefore as follows: Suppose we start from the operators a and $a^\dagger$ satisfying the canonical commutation relations

$$[a, a^\dagger] = I. \tag{2.3}$$

Can we map a and $a^\dagger$ into new operators A and B whose standard commutator satisfies instead the condition suggested by Eq. (2.1), i.e.

$$[A, B] = I + (q-1)N, \tag{2.4}$$

having defined, as usual, $N \equiv a^\dagger a$ (the standard number operator)? In other words, after rewriting Eq. (1.1) in the equivalent form (2.1), we reinterpret *the left-hand side only* as the standard commutator of new operators, here denoted by A and B. By doing so, we are aiming to prove that the standard commutator structure in quantum mechanics can be preserved, while the mathematics of 1-parameter unitary groups makes it possible to achieve a transition from Eq. (2.3) to Eq. (2.4) (see also comments in Section 6).

For this purpose, we consider a pair of invertible operators S and T chosen in such a way that T is unitary and the original commutation relation is no longer preserved. This means that we define

$$A \equiv SaT^{-1}, \tag{2.5}$$

$$B \equiv Ta^{\dagger}S^{-1}, \tag{2.6}$$

which implies that

$$[A, B] = Saa^{\dagger}S^{-1} - Ta^{\dagger}aT^{-1}, \tag{2.7}$$

and eventually, from Eq. (2.3) and the definition of N,

$$[A, B] = I + SNS^{-1} - TNT^{-1}. \tag{2.8}$$

Note that B is not even the formal adjoint of A, since S is not required to be unitary (which will be shown to be sufficient to realize our noncanonical map). Since we require that the commutator (2.8) should coincide with the commutator (2.4), we obtain the equation

$$SNS^{-1} = TNT^{-1} + (q-1)N. \tag{2.9}$$

As said already in Section 1, we are dealing with maps which do not preserve the canonical commutation relations. The nonlinear map

$$a \to \sqrt{\frac{[n]}{n}a}$$

provides an example of such a transformation. Our commutation relations (2.4) are not the same as those of (1.1), for which

$$[n] = \frac{q^n - 1}{(q-1)}$$

but correspond instead to

$$[n] = n + (q-1)\frac{n(n-1)}{2}$$

which is a polynomial deformation.

3. APPLICATION OF THE STONE THEOREM

Having obtained the fundamental Eq. (2.9) we point out that, since T is taken to be unitary, we can exploit the Stone theorem (Stone, 1932), according to which to every weakly continuous, 1-parameter family $U(s)$, $s \in \mathbf{R}$ of unitary operators on a Hilbert space $\mathcal{H}$, obeying

$$U(s_1 + s_2) = U(s_1)U(s_2), \quad s_1, s_2 \in \mathbf{R}, \tag{3.1}$$

there corresponds a unique self-adjoint operator A such that (Prugovecki, 1981; Reed and Simon, 1972)

$$U(s) = e^{isA}, \tag{3.2}$$

for all $s \in \mathbf{R}$. More precisely, the Stone theorem states that, if $U(s)$, $s \in (-\infty, \infty)$, is a family of unitary transformations with the group property (3.1) and such that $(U(s) f, g)$ is a measurable function of s for arbitrary f and g in an abstract Hilbert space, then there exists a unique self-adjoint operator A such that $U(s) = e^{isA}$.

In our problem, we therefore consider a real parameter u and a self-adjoint operator B such that

$$T = T(u) = e^{iuB} \quad u \in \mathbf{R}. \tag{3.3}$$

We exploit Eq. (3.3) after choosing $B = P$ for convenience (see comments below), i.e., the momentum operator canonically conjugate to the position operator Q. In $\hbar = 1$ units, the annihilation and creation operators read

$$a \equiv \frac{1}{\sqrt{2}}(Q + iP), \tag{3.4}$$

$$a^\dagger \equiv \frac{1}{\sqrt{2}}(Q - iP), \tag{3.5}$$

and hence the number operator can be written in the form

$$N \equiv a^\dagger a = \frac{1}{2}(Q^2 + P^2 - I). \tag{3.6}$$

If

$$T(u) \equiv e^{iuP}, \tag{3.7}$$

we can exploit the identities

$$e^{-iuP} Q\, e^{iuP} = Q - uI, \tag{3.8}$$

$$e^{iuP} P = P\, e^{iuP}, \tag{3.9}$$

to obtain

$$TQT^{-1} = Q + uI, \tag{3.10}$$

$$TP = PT, \tag{3.11}$$

and hence

$$\begin{aligned} TNT^{-1} &= \frac{1}{2}TQT^{-1}TQT^{-1} + \frac{1}{2}TPT^{-1}TPT^{-1} - \frac{I}{2} \\ &= \frac{1}{2}(Q + uI)^2 + \frac{1}{2}P^2 - \frac{I}{2} = N + uQ + \frac{u^2}{2}I. \end{aligned} \tag{3.12}$$

It is now clear that the choice $B = P$ in (3.3), although not mandatory, is a matter of convenience, since it makes possible to obtain a manageable expression for TNT^{-1}. This formula, resulting from the particular choice (3.7), can be inserted into Eq. (2.9) which now becomes an equation for the unknown operator S, i.e.

$$SNS^{-1} = qN + uQ + \frac{u^2}{2}I, \tag{3.13a}$$

or also, more conveniently,

$$S(Q^2 + P^2)S^{-1} = q(Q^2 + P^2) + 2uQ + (u^2 - (q-1))I. \tag{3.13b}$$

Now we consider the complete orthonormal set of harmonic oscillator states, denoted by $|n\rangle$ with the abstract Dirac notation. On acting on both sides of (3.13b) with S from the right-hand side one finds

$$S(2N + I) = q(2N + I)S + 2uQS + (u^2 - (q-1))S. \tag{3.14}$$

Since the task of finding S is equivalent to the evaluation of all its matrix elements, we point out that this equation leads to an equation for matrix elements of S upon exploiting the resolution of the identity

$$I = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{3.15}$$

when we write $S = SI$, and defining

$$S_{m,n} \equiv \langle m|S|n\rangle. \tag{3.16}$$

Since $N|m\rangle = m|m\rangle$, while $Q = \frac{1}{\sqrt{2}}(a + a^\dagger)$, one finds, after evaluation of the bra $\langle m|$ on both sides of Eq. (3.14), the equation

$$\begin{aligned}[]&[((2n+1) - q(2m+1) - (u^2 - (q-1)))S_{m,n} \\ &\quad - u\sqrt{2}(\sqrt{m}S_{m-1,n} + \sqrt{m+1}S_{m+1,n})] = 0,\end{aligned} \tag{3.17}$$

where the standard properties $a|m\rangle = \sqrt{m}|m-1\rangle$ and $a^\dagger|m\rangle = \sqrt{m+1}|m+1\rangle$ have been used. Equation (3.17) implies that

$$(2(n - mq) - u^2)S_{m,n} = u\sqrt{2}(\sqrt{m}S_{m-1,n} + \sqrt{m+1}S_{m+1,n}). \tag{3.18}$$

For given values of q and u, this set of equations should be studied for all values of $n, m = 0, 1, \ldots, \infty$. If $mq + \frac{u^2}{2}$ is not an integer, this infinite set yields the matrix element $S_{m,n}$ as a linear combination of $S_{m-1,n}$ and $S_{m+1,n}$, i.e.

$$S_{m,n} = A_{mn}S_{m-1,n} + B_{mn}S_{m+1,n}, \tag{3.19}$$

where

$$A_{mn} = \frac{u\sqrt{2m}}{(2(n - mq) - u^2)} \qquad B_{mn} = \frac{u\sqrt{2(m+1)}}{(2(n - mq) - u^2)}. \tag{3.20}$$

In agreement with our assumptions, these equations show that the operator S is not unitary, since it fails to satisfy the basic condition $SS^\dagger = I$.

To prove the possibility of realizing S as an invertible operator, we find it more convenient to revert to the operator equation (3.14), here written in the form

$$S(Q^2 + P^2) = q(Q^2 + P^2)S + 2uQS + \beta S, \tag{3.21}$$

having defined $\beta \equiv u^2 - (q - 1)$. Here the left- and right-hand sides are operators acting on square-integrable stationary states $\psi(x)$. In one spatial dimension, Q can be realized as the operator of multiplication by x, and P as the operator $-i\, d/dx$, if the coordinate representation is chosen. If S is taken to be the operator of multiplication by an invertible function f, i.e.

$$S : \psi \rightarrow f(x)\psi(x)$$

with $f : x \rightarrow f(x)$ invertible, we have to check that the resulting differential equation for $\psi(x)$ admits square-integrable solutions. Indeed, the choices outlined imply that Eq. (3.21) leads to the following differential equation for $\psi(x)$:

$$\left[\frac{d^2}{dx^2} + \varphi_1(x)\,\frac{d}{dx} + \varphi_2(x)\right]\psi(x) = 0, \tag{3.22}$$

where

$$\varphi_1(x) \equiv \frac{2q}{(q-1)}\frac{f'}{f}, \tag{3.23}$$

$$\varphi_2(x) \equiv -x^2 + \frac{-qf'' + 2uxf + \beta f}{(1-q)f}. \tag{3.24}$$

To ensure that the origin is a regular singular point of Eq. (3.22) we have to choose f in such a way that φ_1 has, at most, a first-order pole at $x = 0$, and φ_2 has at most a second-order pole at $x = 0$. For example, such conditions are fulfilled if $f : x \rightarrow x$, because then φ_1 has a first-order pole at 0, while φ_2 has no poles at all therein, being equal to

$$-x^2 + \frac{2ux + \beta}{(1-q)}.$$

The resulting equation reads

$$\left[\frac{d^2}{dx^2} + \frac{2q}{(q-1)}\frac{1}{x}\,\frac{d}{dx} + \left(-x^2 + \frac{2ux + \beta}{(1-q)}\right)\right]\psi(x) = 0, \tag{3.25}$$

and for it the point at infinity is not Fuchsian, as it happens for the ordinary harmonic oscillator in quantum mechanics.

To sum up, we have shown that Eq. (3.21) is compatible with at least one choice of invertible operator S for which the stationary states are square-integrable on the whole real line (the potential term in Eq. (3.25) being dominated at large x by an even function which diverges at infinity). We have not considered the exponential map as a candidate for f since its inverse, the logarithm, is not defined for negative x, while the ordinary oscillator is studied for all values of x.

4. MODIFIED EQUATIONS OF MOTION

In the investigation of deformed harmonic oscillators it is rather important to check that the equations of motion satisfied by the annihilation and creation operators defined in (3.4) and (3.5), i.e.

$$\left(\frac{d}{dt} + i\right) a = 0, \tag{4.1}$$

$$\left(\frac{d}{dt} - i\right) a^{\dagger} = 0, \tag{4.2}$$

are preserved (Man'ko *et al.*, 1996a). Here, however, we have mapped $(a, a^{\dagger})$ into operators (A, B) whose standard commutator satisfies instead Eq. (2.4). It is therefore not obvious that the equations of motion (4.1) and (4.2) are preserved. Indeed, by allowing for a time dependence of T and S one finds, by virtue of (2.5) and (4.1), that

$$\frac{dA}{dt} = \dot{S}aT^{\dagger} + S\dot{a}T^{\dagger} + Sa\dot{T}^{\dagger} = \dot{S}aT^{\dagger} + S(a\dot{T}^{\dagger} - iaT^{\dagger}). \tag{4.3}$$

This leads to

$$\left(\frac{d}{dt} + i\right) A = \dot{S}aT^{\dagger} + Sa\dot{T}^{\dagger}. \tag{4.4}$$

Now we would like to re-express the right-hand side of Eq. (4.4) in such a way that a is replaced by A. For this purpose, we use Eq. (2.5), the unitarity of T and the invertibility of S to find

$$aT^{\dagger} = S^{-1}A, \tag{4.5}$$

$$Sa = AT, \tag{4.6}$$

and hence the operator A obeys the first-order equation

$$\left(\frac{d}{dt} + i\right) A = \dot{S}S^{-1}A + AT\dot{T}^{\dagger}. \tag{4.7}$$

An analogous procedure shows that

$$\frac{dB}{dt} = \dot{T}a^{\dagger}S^{-1} + T\left(ia^{\dagger}S^{-1} + a^{\dagger}\frac{dS^{-1}}{dt}\right), \tag{4.8}$$

and hence

$$\left(\frac{d}{dt} - i\right)B = \dot{T}a^{\dagger}S^{-1} + Ta^{\dagger}\,\frac{dS^{-1}}{dt} = BS\,\frac{dS^{-1}}{dt} + \dot{T}T^{\dagger}B, \tag{4.9}$$

where we have used the identities

$$Ta^{\dagger} = BS, \tag{4.10}$$

$$a^{\dagger}S^{-1} = T^{\dagger}B. \tag{4.11}$$

5. EQUATIONS OF MOTION VS. COMMUTATORS

In ordinary quantum mechanics one knows, thanks to the work of Wigner (1950) and Man'ko *et al.* (1996b), that the equations of motion do not determine uniquely the commutation relations one relies upon. In our case, this amounts to asking whether, reversing the previous logical order, Eqs. (4.7) and (4.9) are more fundamental than the commutator (2.4), and to which extent a solution of Eqs. (4.7) and (4.9) determines uniquely the commutator of A with B.

Indeed, on defining the first-order operators $\varphi \equiv \frac{d}{dt} + i$ and $\gamma \equiv \frac{d}{dt} - i$, and considering the commutators

$$[A, T\dot{T}^{\dagger}] \equiv C_1, \tag{5.1}$$

$$\left[B, S\,\frac{dS^{-1}}{dt}\right] \equiv C_2, \tag{5.2}$$

Eqs. (4.7) and (4.9) can be written in the form

$$(\varphi - \dot{S}S^{-1} - T\dot{T}^{\dagger})A = C_1, \tag{5.3}$$

$$\left(\gamma - S\,\frac{dS^{-1}}{dt} - \dot{T}T^{\dagger}\right)B = C_2. \tag{5.4}$$

The resulting analysis, far from being of purely formal value, goes at the very heart of the problem: one can solve for A and B upon inverting the operators in round brackets in Eqs. (5.3) and (5.4), and this makes it necessary to find their Green functions. But there may be more than one Green function, depending on which initial condition is chosen. Assuming that such a choice has been made, one can write

$$A = (\varphi - \dot{S}S^{-1} - T\dot{T}^{\dagger})^{-1}C_1, \tag{5.5}$$

$$B = \left(\gamma - S\frac{dS^{-1}}{dt} - \dot{T}T^{\dagger}\right)^{-1} C_2, \tag{5.6}$$

and their commutator is not obviously equal to (see (2.4))

$$I + (q-1)a^{\dagger}a = I + (q-1)T^{\dagger}BAT$$

where we have inverted Eqs. (2.5) and (2.6) defining A and B to find

$$a = S^{-1}AT, \qquad a^{\dagger} = T^{\dagger}BS.$$

6. CONCLUDING REMARKS

Starting from Eqs. (1.1) and (2.1) we have pointed out that deformed commutators can be "replaced" by a map of the standard commutation relations (2.3) into the modified form (2.4). As far as we can see, this is by no means equivalent to deformation quantization. Our effort to build such a map reflects instead the desire to preserve the standard commutator structure, while using some basic mathematical tools to prove that the map of Eq. (2.3) into Eq. (2.4) is feasible. This leads to the introduction of two different invertible operators S and T with T unitary, subject to the consistency condition (2.9). From the point of view of ideas and techniques, this is the original contribution of our paper. Section 3 proves that a careful use of the Stone theorem makes it possible to fulfill such a condition with S invertible, while Sections 4 and 5 have studied how the equations of motion are modified, and what sort of correspondence exists between them and the commutator (2.4).

Our framework can be made broader by studying the case when neither S nor T is unitary (see (2.5) and (2.6)), but we see no (obvious) advantage in doing so. Our investigation is of interest for the mathematical foundations of quantum mechanics because *it shows under which conditions it is possible to avoid deforming the composition law of classical observables* (cf. Bayen *et al.*, 1978; Biedenharn, 1989; Bozejko *et al.*, 1997; Jorgensen and Werner, 1994; MacFarlane, 1989; Sternheimer, 1998; Sun and Fu, 1989; Werner, 1995). Further developments can also be expected, because the link between the superoperator formalism (Los, 1978) and the maps defined by our Eqs. (2.5) and (2.6) deserves a thorough investigation.

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